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ABSTRACT

This chapter examines the scope of the mathematical content educators expect students to understand after they have participated in mathematics courses. It is organized under four headings: (1) Traditional School Mathematics, to clarify what the shift is away from; (2) Mathematics as Human Activity, to portray the direction the shift is toward; (3) Mathematics Worth Teaching; and (4) Speculations about School Mathematics in the Future. (SOE)

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Mathematics Worth Teaching, Mathematics Worth Understanding

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When authors in this book describe classrooms that promote mathematical understanding, they each attempt to answer the question: What mathematics is it that students are expected to understand? The answers these authors give, however, may surprise some readers because although the titles of the mathematical domains described—number, geometry, quantities, statistics, and algebra—are familiar, the approach to discussing them is not. Rather than providing a list of specific concepts and skills to be mastered as has often been done in the past, each author focuses on a few key ideas in a domain, the interconnections between them, and examples of how students can grasp an understanding of the ideas as a consequence of exploring problem situations. The descriptions of mathematical content in this book are based on the epistemological shift in expectations about school mathematics that has emerged during the past decade; the chapters reflect attempts to redefine content in light of that shift.

The purpose of this chapter is to examine the scope of the mathematical content we expect students to understand after they have participated in the mathematical curricula described. This chapter has been organized under four headings: (a) traditional school mathematics, to clarify what the shift is away from; (b) mathematics as a human activity, to portray the direction the shift is toward; (c) mathematics worth teaching, to provide an overview to the answers provided in the following chapters to the initial question (What is it that students are expected to understand?); and, finally, (d) speculations about school mathematics in the future.

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TRADITIONAL SCHOOL MATHEMATICS

Mathematics is perceived by most people as a fixed, static body of knowledge. Its subject matter includes the mechanistic manipulation of a variety of numbers and algebraic symbols and the proving of geometric deductions. This perception has determined the scope of the content to be covered and the pedagogy of the school mathematics curriculum. Specific objectives, which students are to master, have been stated; the teacher's role has been to demonstrate how a manipulation is to be carried out or to explain how a concept is defined; and students have been expected to memorize facts and to practice procedures until they have been mastered.

The arithmetic of whole numbers, fractions, decimals, and percents represents the majority of the first 7 of 8 years of the school mathematics curriculum for all students. For those students who successfully complete the hurdles along the arithmetic path, yearlong courses in algebra and geometry follow. Then, for many college-bound students, what follows is a second-year course of algebra and perhaps a year course of precalculus mathematics. Finally, for a few students, a course in calculus is often available. In each course, the emphasis is on mastering a collection of fixed concepts and skills in a certain order. This layer-cake structure that we have inherited is deeply embedded not only in our curricular structure but in our larger society's expectations regarding schools. This approach to mathematics education reinforces the tendency to design each course primarily to meet the prerequisites of the next course. This layer-cake filtering system is responsible for the unacceptably high attrition from mathematics that plagues our schools.

The traditional three-segment lesson, which has been observed in many classes, involves an initial segment where the previous day's work is corrected. Next, the teacher presents new material, often working one or two new problems followed by a few students working similar problems at the chalkboard. The final segment involves students working on an assignment for the following day.

This mechanistic approach to instruction of basic skills and concepts isolates mathematics from its uses and from other disciplines. The traditional process of symbol manipulation involves only the deployment of a set routine with no room for ingenuity or flair, no place for guesswork or surprise, no chance for discovery; in fact, no need for the human being. Furthermore, the pedagogy of traditional instruction includes a basal text (which is a repository of problem lists), a mass of paper-and-pencil worksheets, and a set of performance tests. There is very little that is interesting to read. Workbook mathematics thus gives students little reason to connect ideas in today's lesson with those of past lessons or with the real world. The tests currently used ask for answers that are judged right or wrong,

but the strategies and reasoning used to derive answers are not evaluated. This portrayal of school mathematics—a tedious, uninteresting path to follow, with lots of hurdles to clear—bears little resemblance to what a mathematician or user of mathematics does. What is clear is that students do not do mathematics in traditional school lessons. Instead, they learn a collection of techniques that are useful for certain purposes. Because mastery of techniques is defined as knowledge, the acquisition of those techniques becomes an end in itself, and the student spends his or her time absorbing what other people have done.

Traditional school mathematics has failed to provide students with any sense of the importance of the discipline's historical or cultural importance, nor any sense of its usefulness. Is it any wonder that many students dislike mathematics and fail to learn it? The premise of this book is that traditional teaching and learning of mathematics has not enabled students to learn mathematics with understanding and that our first step must be to redefine mathematics.

MATHEMATICS AS A HUMAN ACTIVITY

School mathematics should be viewed as a human activity that reflects the work of mathematicians—finding out why given techniques work, inventing new techniques, justifying assertions, and so forth. It should also reflect how users of mathematics investigate a problem situation, decide on variables, decide on ways to quantify and relate the variables, carry out calculations, make predictions, and verify the utility of the predictions. Underlying this perspective of human activity is our use of Thurston's (1990) metaphor of a tree to describe the discipline of mathematics:

> Mathematics isn't a palm tree, with a single long straight trunk covered with scratchy formulas. It's a banyan tree, with many interconnected trunks and branches—a banyan tree that has grown to the size of a forest, inviting us to climb and explore. (p. 7)

For school mathematics, this vision emphasizes human actions, "climbing and exploring." Thurston's image of mathematics as the everspreading banyan tree reflects the notion that mathematics is a plural noun in that there are several intertwined trunks or branches (strands or domains). Furthermore, this image extends to the tree's ever-widening root system, drawing in the wide range of activities, interests, linguistic capability, kinesthetic sense, and informal knowledge that students bring to us. It is this broadly inclusive metaphor of mathematics, and the exploration of problems as a way of learning mathematics, that underlies all the subject matter examples of this book.

The discipline of mathematics involves a vast assemblage of ideas in several related content domains and is defined by the community of mathematicians, mathematics educators, and users of mathematics (Hersh, 1997). The content domains have been derived via a continually evolving, culturally shared science of patterns and languages, which is extended and applied through systematic forms of reasoning and argument. Mathematics is both an *object* of understanding and a *means* of understanding. These patterns and languages are an essential way of understanding the worlds we experience—physical, social, and even mathematical. Mathematics is ever alive—as alive as any branch of science today—and is neither static nor fixed in time, but changing. Because mathematics evolves, so also must the school curriculum—the educational organization and sequencing of that content.

The emerging redefinition of school mathematics is based on an epistemological shift in perspective regarding what is important to students to know and understand and is reflected in curriculum documents from several countries (e.g., Australia, Japan, the United Kingdom), as well as from the United States. The problem being faced is that there has been a conflict between school mathematics as it has been organized and taught and the emerging perspective about mathematics. To provide even a sketch of a content outline or sequence for K through 12 mathematics that reflects this epistemological shift is well beyond the scope of this chapter and this book. Instead, our aim is to provide a background to the choices about what mathematics to teach (for understanding) from the many choices that are or will become available. Note that in contrast to the traditional classroom, doing mathematics from this perspective cannot be viewed as a mechanical performance or an activity that individuals engage in solely by following predetermined rules.

Curriculum activities that reflect this perspective are those that involve students in problem solving and that encourage mathematization. Such tasks include situations that are subject to measure and quantification, that embody quantifiable change and variation, that involve specifiable uncertainty, that involve our place in space and the spatial features of the world we inhabit and construct, and that involve symbolic algorithms and more-abstract structures. In addition, they encourage the use of mathematical languages for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing. These systems of signs and symbols extend the limited powers of the human mind in many directions, and they make possible a long-term (cross-generational) cultural growth of the subject matter. Finally, such situations embody systematic forms of reasoning and argument to help establish the certainty, generality, consistency, and reliability of the individual's mathematical assertions.

If mathematics is to serve students' needs to make sense of experience arising outside of mathematics instruction and mathematics itself, includ-

ing making sense in the various sciences, it must be firmly rooted in and connected to that experience. And its systems of signs and symbols must be learned and experienced as genuine, functioning languages—for expressing, communicating, reasoning, computing, abstracting, generalizing, and formalizing—that the student experiences as serving his or her real needs. Similarly, the systematic logical forms of reasoning and argument must be learned through their satisfying personally and socially experienced needs for certainty and reliability—for establishing, for the student, what is true and what is not true.

In summary, there are several ways of viewing the discipline and choices to be made about which aspects of mathematics are to be included in the school curriculum, but the patterns, signs, symbols, and rules from certain branches of contemporary mathematics should be learned by all students and ideas from other branches known by some students. Furthermore, students should not only know the concepts and procedures for some parts of mathematics but also understand how mathematics is created and used.

This view of mathematics is, above all, integrative: It sees everything as part of a larger whole, with each part sharing reciprocal relationships with other parts. It stresses the acquisition of understanding by all (including the traditionally underprivileged), to the highest extent of their capability, rather than the selection and promotion of an elite. It is a philosophy that simultaneously stresses erudition and common sense, integration through application, and innovation through creativity. Most important, it stresses the creation of knowledge. Against this broad and ambitious view of mathematics, traditional school mathematics appears thin, lifeless, and isolated.

If students follow our approach to mathematical content, we believe that they will learn to formulate problems and develop and apply strategies to find solutions in a range of contexts. By exploring problems, they will learn to verify and interpret results and generalize solutions. In so doing, they will learn to apply mathematical modeling and become confident in their ability to address real-world problem situations. If they reason through their problem situations, students will develop the habit of making and evaluating conjectures and of constructing, following, and judging valid arguments. In the process, they will deduce and induce; apply spatial, proportional, algebraic, and graphical reasoning; construct proofs; and formulate counterexamples. In this sense, problem solving should be the basis of the mathematics curriculum.

MATHEMATICS WORTH TEACHING

Because students are more likely to learn mathematics in the context of problem situations, the work of the National Center for Research in Mathematic Sciences Education (NCRMSE) Principal Investigators focused on

curricular ideas organized in a vertical strand structure of mathematics. There are two aspects to this work. The first involved specifying the scope of the mathematics in each strand that students are expected to know and understand, and the second involved outlining the characteristics and sequence of instructional activities that would engage students in learning mathematics with understanding. Our initial step in specifying the content was to identify the content domains (or strands) that society expects all students to learn with understanding. The domains we examined were number, quantities, algebra, geometry, and statistics. The domains were selected because of their generality and ability to subsume more specialized components of the discipline.

The second step involved identifying the big ideas in each domain that are worthy of extended time and effort. These big ideas are important for students to find, discover, use, or even invent for themselves. For example, the initial related concepts of addition and subtraction of whole numbers (a substrand of the whole-number strand) involve:

- symbolic statements that characterize the domain (e.g., $a + b = c$ and $a - b = c$, where a, b, and c are natural numbers);
- the implied task (or tasks) to be carried out (for addition and subtraction, this may involve describing the situations where two of the three numbers [a, b, and c] in the statements are known and the third is unknown);
- rules that can be followed to represent, transform, and carry out procedures to complete the task (e.g., "Find the unknown number using one or more of such procedures as counting; basic facts; symbolic transformations, such as $a + [\ ] = c \leftrightarrow c - a = [\ ]$; and place value [e.g., $345 = 3\ (100) + 4\ (10) + 5\ (1)$]);
- computational algorithms for larger numbers (e.g., adding 34 and 28 involves finding the sums of $30 + 20$, $4 + 8$; regrouping 12 into $10 + 2$, and finding the sum of $30 + 20 + 10 + 2$);
- the set of situations that have been used to make the concepts, the relationships between concepts, and the rules meaningful (e.g., join-separate, part-part-whole, compare, equalize).

Note that the set of situations that give meaning to the concepts and rules are considered equally as important as learning to follow the procedural rules. In fact, it is assumed that they are the means by which the procedural rules are understood. The structure of each strand also acknowledges that the big and powerful ideas and connections between them require years to develop and are understood in a variety of forms. Thus, the big ideas are stressed for all students throughout the school years rather than at the end for an elite minority.

From this example, it should be clear each strand comprises several closely related substrands. For example, the whole-number strand includes assigning numbers to sets or objects by counting or measuring, understanding place value, adding and subtracting, multiplying and dividing, and exploring the properties of such numbers (e.g., evenness and primeness). The big ideas in each substrand must be identified as was outlined earlier for addition and subtraction. Such detailed specifications describe the scope of the mathematical topics and ideas students should be expected to know. Understanding these ideas, however, involves relating a key idea to other ideas, explaining and justifying relationships, and so forth. Furthermore, the key ideas in different strands are closely interrelated. For example, measuring lengths, areas, volumes, angles, distances, and so on are important aspects of geometry.

Thus, to sequence instruction based on the tenets of mathematics as a human activity implies that tasks cannot be organized in a particular sequence to cover each detail, as is done in traditional curricula. Instead, instructional units or tasks that focus on investigation of problem situations need to be created. From such situations, the key ideas in each strand can be acquired or constructed and related to other ideas in other strands or substrands, and the uses of these ideas in other disciplines identified. The criteria for the selection of such tasks involve answering the following five questions:

Do the Tasks Lead Anywhere? To answer this question involves extending the map of the domain by beginning with the prior knowledge that students have and then developing instructional tasks that allow students to proceed gradually from their informal knowledge of the ideas in the domain to more formal notions. Note, however, that mapping domains is not an easy task because it involves forming a network of multiple possible paths and is not partitioned into discrete segments. It should also be noted that there is no one correct sequence of concepts and activities in any domain.

Again an important aspect of this step is that learning with understanding in any domain takes time. Proceeding from informal ideas to a more formal understanding of a domain may take several years, with periodic review and extension of prior notions. To accomplish this, instructional tasks need to be designed, organized with other activities, and tentatively sequenced, so that students have an opportunity to investigate increasingly complex problem situations within (and often across) domains. For example, for students to learn how to add and subtract with understanding, we might start with a variety of problem situations involving one-digit numbers where students could either model or count to find the answers. Later, students could solve problems in similar situations involving multidigit numbers where modeling and counting are inefficient, which

would enable them to invent and learn to use computational algorithms. In addition, tasks should not focus only on the ideas in a single strand or substrand, but rather break through the artificial barriers between mathematical topics.

The selection of tasks is not an easy task. Although there is no doubt that many interesting tasks exist or can be created, "Do they lead anywhere?" is a serious question. Romberg (1992) stated that:

> Too often a problem is judged to be relevant through the eyes of adults, not children. Also, this perception is undoubtedly a Western, middle-class, static vision. Concrete situations, by themselves, do not guarantee that students will see relevance to their worlds, they may not be relevant for *all* students nor prepare them to deal with a changing, dynamic world. (p. 778)

For example, Hatano and Inagaki (1998) described an instructional episode with first-grade children in a Japanese classroom. The students familiar with join-separate problems were presented with the comparison problem, "There are 12 boys and 8 girls. How many more boys than girls are there?" Most children answered correctly that there were four more boys, but one child insisted that subtraction could not be used because it was impossible to subtract girls from boys. Furthermore, none of the students who had answered correctly could offer a persuasive argument against this assertion. It was only after the students physically modeled the situation that they realized that finding the difference involved subtracting the 8 boys who could hold hands with girls from the 12 boys (the total number of boys).

Do the Tasks Lead to Model Building? One way of making situations meaningful is to expect students to represent phenomena by means of a system of theoretically specified objects and relations (i.e., build a model such as boys and girls holding hands). Meaningful inquiry, involving cycles of model construction, model evaluation, and model revision, is central to the professional practice in mathematics but is largely missing from school instruction. Modeling practices are diverse, ranging from the construction of physical models to the development of abstract symbol systems.

In fact, any instructional unit should begin with an empirical situation, engaging to students, that presents a problem for which an answer is sought. However, the use of such words as *problem*, *answer*, and *engaging* can be misleading. Because real situations rarely appear well defined, the identification of a problem amenable to mathematical treatment is often difficult and involves many skills not related to mathematics. Nevertheless, the situation needs to be worthy of investigation (engaging). Finding a solution involves determining the essential or significant features of the

situation, translating these features into a mathematical model, validating the model, and interpreting the results, usually in multiple cycles. Note that the task of problem solving is never complete when a mathematical answer is obtained. The mathematical results must be reinterpreted in terms of the initial problem situation. Furthermore, modeling emphasizes a need for forms of mathematics that are typically underrepresented in the standard curriculum, such as spatial visualization and geometry, discrete structures, and measures of uncertainty. In particular, incorporating the practice of modeling in instruction allows for the further exploration and identification of important big ideas in a domain.

Do the Tasks Lead to Inquiry and Justification? The identification process in mathematical modeling involves an attempt to sort out the essential or significant features of the situation. This simplification or idealization is a crucial stage because the general problem is often exceedingly complex, involving many processes. Some features will appear significant; many, irrelevant. Simplification is obtained when, by ignoring insignificant features, an originally complex problem is idealized to one that is mathematically tractable. Note that the history of the discipline is filled with instances of mathematicians inventing a way of representing an important feature of a problem (e.g., Descartes' invention of a way of describing the location of a fly on his bedroom ceiling [a point on a plane surface]). Although students cannot be expected to reinvent all mathematics, they should be expected to invent routines, formulas, or expressions as a consequence of their investigations. Furthermore, as they create such assertions they should be expected to justify them in some form. Some form of validation is usually carried out throughout the formulation of a model (i.e., the formulas or other mathematical relations set up in the model are continually checked with the initial situation). Also, the mathematics used in the model must be self-consistent and obey all the usual laws of mathematical logic.

With appropriate guidance from teachers, a student's informal models can evolve into models for increasingly abstract mathematical reasoning. The development of ways of symbolizing problem situations and the transition from informal to formal semiotics (called progressive formalization) are important aspects of these instructional assumptions.

As part of each activity, students should develop an appreciation for the evidential grounds for a claim and learn to evaluate those grounds by having opportunities to participate in genuine mathematical inquiry in school. By genuine inquiry, we mean the process of raising and evaluating questions grounded in experience, proposing and developing alternative explanations, marshaling evidence from various sources, representing and presenting that information to a larger community, and debating the persuasive power of that information with respect to various claims. In mathe-

matics, for example, students are routinely taught computational algorithms but rarely are they asked to invent their own strategies for solving problems and to discuss why those strategies work. The information presented by authors in the next section demonstrates that students can invent sophisticated procedures for solving problems, operating with numbers, and reasoning about space and geometry. Beyond its contribution to students' identities as effective learners in a discipline, genuine inquiry also provides them with practice in and eventual understanding of the criteria for evidence and explanation patterns characteristic of mathematics.

Do the Tasks Involve Flexible Use of Technologies? As instructional tasks are created, we must add increased flexible use of new technologies to support the learning environments. Rather than just providing paper-and-pencil tasks for students to do independently at desks, we must provide tasks and materials that allow students to explore a domain, often solving problems as a group. The technology may also include video recordings, calculators, and computers as student tools. Evaluation involves judgments by students and teachers of the coherence of presentations, the reasoning given, and so on. Note that simulations and computer-based models have proven to be the most powerful resources for the advancement and application of mathematics and science since the origins of mathematical modeling in the Renaissance. As suggested later in this chapter, the move from static models (built in inert media) to dynamic models and the associated visualization and analytic tools (built in interactive media) is profoundly changing not only the nature of mathematics but the nature of inquiry in the discipline itself.

Are the Tasks Relevant to Students? To be relevant, an instructional activity should intrinsically motivate students to seek information so they can make sense of a problem situation, find satisfactory explanations for the validity of a given rule, and be able to justify the success of a procedure. Thus, the tasks should include expecting students to generate inferences, to check their plausibility, and to coordinate pieces of old and new information to build an enriched and coherent model or representation of the problem situation. The depth of knowledge of a student in any domain comes from structured experiences that promote personal ownership of the ideas in that domain. Unfortunately, what motivates a student may vary from individual to individual. What is clear is that the tasks should build upon what students know, produce cognitive incongruity, create some curiosity to resolve the incongruity, and so forth. But at the same time, a task cannot be too challenging (produce too much cognitive conflict), causing students to dismiss the activity as impossible or meaningless.

SPECULATIONS ABOUT SCHOOL MATHEMATICS IN THE FUTURE

During the past quarter century, Thurston's everspreading banyan tree has been given very powerful fertilizer. On that tree of mathematics, new branches have emerged and grown rapidly; many old branches have been revitalized; and some branches have withered. As Valentine, a boisterous young mathematical biologist in Stoppard's (1993) play *Arcadia*, noted; "The future is disorder. A door like this has cracked open five or six times since we got up on our hind legs. It's the best possible time to be alive, when almost everything you thought you knew is wrong" (p. 48).

Although this is a bit of theatrical hyperbole, he was pointing to something profoundly important in the evolution of science and mathematics that will grow with increasing force in the next century: the iterative mathematics now possible in the computational medium and the nonlinear science that it makes possible. Later this chapter, we examine aspects of this evolution not to provide an introduction to this explosively growing way of approaching mathematics and science but to orient the reader first to the fact that mathematics continues to change even more rapidly than ever before and, second, to the directions in which this change is occurring.

But first we examine the depth of the change from past schooling practices to the practices concretely illustrated by the authors in the following chapters. Previously, students studied number for number's sake, or algebra for algebra's sake, and later applied what they had learned to solve problems and perhaps even to engage in serious mathematical modeling. We suggest the reverse: that number, algebra, and most other core school mathematics should arise in the service of making sense of individual experience. Furthermore, as the content chapters illustrate very powerfully, students are capable of a significant amount of mathematics at an early age. Because mathematics is a living, dynamic discipline, further changes in school mathematics are inevitable.

We now illustrate the direction of future work briefly examining nonlinear dynamic systems. Since the explosion in nonlinear mathematics and science in the 1970s, nonlinear dynamic systems have transformed the sciences—physical, social, biological, neuropsychological—and much of analytic mathematics. Classical mathematics and science are now seen to provide only a very narrow, idealized view of phenomena compared to the new nonlinear models. Consider, for example, classical models of the motion of a pendulum. They were accurate only if the pendulum was assumed to be a point-mass suspended by a weightless, frictionless string, that, most importantly, swung through only a very small angle about the vertical from a stationary support. Imagine that you are holding a string from which a pencil is hanging. You are swinging the pencil by moving you hand

from side to side. This situation violates all of these conditions but can be modeled as a nonlinear dynamic system. Moreover, the mathematics that results includes much of the traditional mathematics as special, simplified cases.

Computer technology plays twin roles in this explosion of new mathematics and science by providing iterative numerical solutions to nonlinear difference and differential equations that are not solvable analytically (i.e., they do not have solutions as formulas) and graphical depictions of the states of the systems being modeled. The resulting models stretch our notions of randomness and determinism. For example, things can be both deterministically defined—indeed, quite simply defined—yet have chaotic and unpredictable longer-term behavior. We are also learning how complex systems of all sorts—the brain, economies, fluid flows, epidemics, biological ecological systems, chemical reactions, and so on—can be self-organizing.

The consequences of these developments are, first, that most dynamic situations in the world are now recognized to be nonlinear, not linear; and, second, that the simplest of nonlinear systems can embody an extraordinary amount of complexity. Closed-form algebraic solutions are rare, and most solutions are iterative, numeric, and graphic. This fact has profound implications for our ability to use mathematics to predict the behavior of nonlinear dynamic systems. For example, no matter how accurate our data or how powerful our computers, it is difficult to improve weather-forecasting accuracy beyond 4 days. To get a 5-day accuracy matching today's 4-day accuracy rate (about 80%) is a large and expensive undertaking that involves getting much more data about the atmosphere around the world and even more powerful computers than the fastest available supercomputers now used to model the atmosphere. Similar constraints apply wherever we turn because, as we are learning, most phenomena in the world are the result of nonlinearly interacting systems.

As already noted, the second factor associated with technology's role in this kind of mathematics is its graphical ability to display states of the system being modeled. These graphical images reveal astounding visual complexity and beauty that contain mixes of regularities and chaotic phenomena. They are usually in two dimensions and usually in color (which acts as a third dimension to display variation). The kinds of analyses that take place in such contexts are highly visual: Patterns and shapes play an important role whereas the ability to manipulate complicated symbol strings plays a reduced role. In a deep sense, the beauty of mathematics becomes available in a new way, and our ways of working with the mathematics are deeply changed (see Stewart, 1990). A whole new set of ideas becomes important in such contexts. For example, although once regarded as relevant only to elite students, topological ideas (e.g., issues of closeness and separability of points, boundaries of regions, open and closed sets, orbits, accumulation points, limits) and point-set topology (once treated

as an abstract specialty) have come alive again. (They were invented a century ago to help make sense of dynamical systems studied more abstractly than they are today.)

These examples concern the mathematics of growth and change, which includes the ideas of traditional calculus as a subset. Long-held assumptions about who can learn what mathematics, and in what order, are being directly challenged by a variety of efforts currently underway. For example, the SimCalc Project (Kaput & Roschelle, in press) is building a mix of technologies and curricula intended to democratize access to the basic ideas underlying calculus beginning in the middle-school grades and even earlier; the Math in Context Core Plus Curriculum high school materials (Coxford et al., 1997) include iterative mathematics as a 4-year strand; the continuing work of Devaney and colleagues (see Devaney, 1998) shows that the ideas of chaos, fractals, and nonlinear mathematics are not only learnable by students as early as middle school but serve as enticing gateways to other mathematics such as linear algebra historically reserved for a small segment of much older students. These examples are but the tip of a newly emerging iceberg. Dynamic geometry illustrates another class of examples in another mathematical strand. The geometry involved is not Euclidean geometry but a new kind of mathematics. And the kinds of thinking and understanding that it requires are subtly different in important ways from that required in traditional geometry. Similarly, although computational techniques in statistics and probability have long been essential, many new forms of statistics have appeared in recent years, evolving, particularly, from resampling techniques. These new forms, sometimes referred to as bootstrapping techniques, depend directly on the computer medium and greatly extend the amount of information that can be extracted from even small sets of data. Indeed, the phenomenon of new mathematics growing in this fertile new computational medium is increasingly common. Old branches of mathematics are growing rapidly as well, and new ones are sprouting. No longer can we assume that the content of school mathematics worth understanding is fixed. The new mathematics, and newly available old mathematics, are becoming a vital part of the intellectual needs of all of our students. They are not merely the intellectual province of a few knowledge elite.

CONCLUSION

The study of mathematics by all students is important, but society's perception of the scope of content that students are expected to understand is changing, as is the field of mathematics itself. These changes make it imperative that any answer to the question "What mathematics is worth teaching?" be periodically considered. No longer can we assume that mathematics is a fixed body of concepts and skills to be mastered.

Nevertheless, regardless of the specific content, the aims of mathematics teaching can be described, in practical terms, as teaching students to use mathematics to build and communicate ideas, to use it as a powerful analytic and problem-solving tool, and to be fascinated by the patterns it embodies and exposes. In this chapter, we have asked the reader to think of mathematics as an ever-expanding banyan tree that invites students to climb and explore. We have emphasized that instructional tasks should be relevant and should invite exploration that leads to further exploration, model building, inquiry, justification, and flexible use of technology. Equally important is the learning of attitudes and habits; the sense of mathematics as a creative process requiring imagination, initiative, and flexibility; and the habits of working systematically, whether independently or cooperatively. Finally, and most important, the learning of mathematics should be an experience from which students derive enjoyment and earn confidence.

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